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ON THE MARTIN BOUNDARY OF RIEMANNIAN PRODUCTS

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Abstract

We describe the minimal Martin boundary of a Riemannian product $X = X_1 \times X_2$ where the factors are complete manifolds with Ricci curvature bounded below. As a consequence we obtain a splitting result for bounded harmonic functions.

0. Introduction

The goal of this paper is to prove a splitting theorem for positive harmonic functions on a Riemannian product, under very general assumptions: we require only that each factor be a complete, noncompact Riemannian manifold with Ricci curvature bounded below.

Given a complete, noncompact Riemannian manifold X, denote by $\lambda_0(X) \leq 0$ the supremum of the closed L^2 spectrum of the Laplace-Beltrami operator on X. For each $\lambda \geq \lambda_0$, the eigenvalue problem

$$\Delta\varphi = \lambda\varphi$$

has positive solutions. For $\lambda > \lambda_0$, $\Delta - \lambda$ is coercive, has a Green function $G^\lambda(x, y) > 0$, and the λ-eigenfunctions on an open set $U \subset X$ define a Brelot harmonic sheaf (for proofs of these facts, see [1], [20]). For each $\lambda \geq \lambda_0$, denote by $\mathcal{M}_1^\lambda(X)$ the space of minimal positive λ-eigenfunctions:

$$\mathcal{M}_1^\lambda = \{0 < f \in C^\infty(X) | \Delta f = \lambda f,\ 0 < g \leq f, \Delta g = \lambda g \Rightarrow g = (\text{const})f\}.$$

Theorem. *Let $X = X_1 \times X_2$ be a Riemannian product, where X_1 and X_2 are complete, noncompact, with Ricci curvature bounded below. Then the following hold.*

(i) *Each minimal positive harmonic function f on X splits as a product*

$$f(x) = K^{\lambda_1}(x^1)K^{\lambda_2}(x^2),$$

where $\lambda_i \geq \lambda_0(X_i)$, $K^{\lambda_i} \in \mathcal{M}_1^{\lambda_i}(X_i)$ for $i = 1, 2$, and $\lambda_1 + \lambda_2 = 0$.

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(ii) *Conversely, each product as above is a minimal positive harmonic function on* X.

As a corollary, we have a "strong harmonicity" result for *bounded* harmonic functions:

Corollary. *Let* $X = X_1 \times X_2$ *as above. Every bounded harmonic function* f *on* X *is strongly harmonic, that is, satisfies*

$$\Delta_1 f = \Delta_2 f = 0.$$

For convenience we state and prove the theorem for two factors; clearly this implies the result for any finite number of factors.

Example 1. Suppose $\mathrm{Ric}(X_1) \geq 0$. Then the corollary and Yau's Liouville theorem [25] imply that if $f(x_1, x_2)$ is a bounded harmonic function on $X = X_1 \times X_2$ (for any X_2 as in the theorem), we actually have $f = f(x_2)$.

Example 2. Let X = universal cover of V, where $-a^2 \leq$ sectional curvatures $(V) \leq 0$ and V has finite volume. By Theorem 1.8 in [5] in the de Rham splitting of X,

$$X = X_1 \times \cdots \times X_r,$$

the irreducible factors X_i are either $\mathbf{R}^n$, symmetric spaces of noncompact type, or "rank-one" manifolds. Our theorem reduces the identification of the Martin boundary $\mathcal{M}_1(X)$ and the Poisson boundary $\Pi(X)$ to the same problem for each factor. This is trivial for $\mathbf{R}^n$, and classical in the case of symmetric spaces (see [13] or [14]). If the "rank-one" factors are actually of strictly negative curvature $(-a^2 \leq \mathrm{sect}(X_a) \leq -b^2,\ b > 0)$, the results of [2], [1] imply that

$$\mathcal{M}_1^\lambda(X_a) \sim \Pi(X_a) \sim X_a(\infty)$$

($X_a(\infty) =$ the "sphere at infinity" of X_a) for any $\lambda > \lambda_0(X_a)$. In the case of general "rank-one" factors our knowledge of $\mathcal{M}_1^\lambda(X)$ is still very incomplete, even for $\lambda = 0$. This example was the original motivation for this work.

The classical example of a description of the minimal Martin boundary in the context of complete manifolds is the work of Karpelevič on symmetric spaces of noncompact type [14]; as a consequence he obtains a proof of Furstenberg's strong harmonicity of bounded harmonic functions. Our proof proceeds by steps similar to [14], but without the advantage of a large isometry group. More recent results on splitting of minimal positive solutions of elliptic and parabolic equations with periodic coefficients have been obtained by Y. Pinchover [21] and Korányi and Taylor

[15]. The main idea to obtain these results (as well as ours) is to use a group of bounded isometries and a suitable Harnack inequality to conclude that minimal positive solutions "split"; in the parabolic case, the role of "bounded isometry" is played by the translation semigroup along the t-axis. It is interesting to observe that, by a result of J. Wolf, the de Rham splitting of a Hadamard manifold which admits bounded isometries contains a nontrivial Euclidean factor [24]. This suggests that the investigation of the irreducible cocompact ("rank-one") nonpositive curvature case will have to proceed via a different method.[1]

J. C. Taylor has recently obtained a result similar to ours, by probabilistic methods [23].

The first two sections of this paper are largely expository: we outline the proof of an integral representation formula for positive solutions of the heat equation on a complete "Riemannian halfspace", based on results from abstract potential theory. In particular, we show that minimal positive solutions of the heat equation may be obtained as limits of normalized heat kernels (the parabolic "Martin boundary" construction). This is no doubt known to experts, but we have not been able to find a concise presentation in the literature. §§3 and 4 contain the proof of the main theorem: starting from a characterization of minimal positive solutions of the heat equation (due to Korányi and Taylor [15]) we obtain a representation formula for positive harmonic functions on a product, which easily implies part (i); part (ii) is proved by appealing to a lemma of Cheng and Yau [6]. In the last section we explain how a standard argument in potential theory implies the corollary.

1. Parabolic Martin boundary of a Riemannian halfspace

In this section we describe the construction of the Martin boundary associated to the heat equation on a Riemannian 'halfspace':

$$\mathcal{H} = X \times (-\infty, 0],$$

where X is a complete, noncompact smooth Riemannian manifold without boundary. The presentation draws on results from abstract potential theory. We are interested in the (Bauer) harmonic space $(\mathcal{H}, P)$, where P is the harmonic sheaf:

$$P(U) = \{u(\overline{x}) \in C(U)|_{u_{|\mathcal{H}'}} \in C^2(U \cap \mathcal{H}') \text{ and } \Delta u = \frac{\partial u}{\partial t} \text{ in } U \cap \mathcal{H}'\}$$

[1] **Added in proof.** In a recent paper, W. Ballmann has shown that the sphere at infinity is regular for the Dirichlet problem in the compact rank-one case (see [4]).

for any $U \subset \mathcal{H}$ open, where $\mathcal{H}' = X \times (-\infty, 0)$ and $\overline{x} = (x, t)$. By 'parabolic function on U' we shall mean $u(x, t) \in P(U)$. A domain $D \subset \mathcal{H}$ of the form $D = B \times I$, where $B \subset X$ is a geodesic ball and $I \subset (-\infty, 0)$ is an open interval, will be called a *cylinder*. It is well known that the parabolic Dirichlet problem is solvable for continuous functions defined on the 'parabolic boundary' of D: $\partial_p D = \partial B \times [s, t] \cup \overline{B} \times \{s\}$. A *superparabolic function* $u(x, t)$ is a lower semicontinuous function with values in $(-\infty, +\infty]$, finite on a dense subset of $\mathcal{H}$, and such that for any cylinder $D \subset \mathcal{H}$,

$$v \text{ parabolic on } D, \quad v \leq u \text{ on } \partial_p D \Rightarrow v \leq u \text{ on } D.$$

u is *subparabolic* if $-u$ is superparabolic; a nonnegative *potential* on $\mathcal{H}$ is a nonnegative superparabolic function whose greatest parabolic minorant is zero; the *support* of a potential is the complement of the largest open set on which it is parabolic.

Let $p(t, x, y)$ be the minimal positive fundamental solution of the heat equation on X. The natural Green function associated with $(\mathcal{H}, P)$ is

$$\dot{G}(\overline{x}, \overline{y}) = \begin{cases} p(t-s, x, y), & t > s, \\ 0, & t \leq s, \end{cases}$$

where $\overline{x} = (x, t) \in \mathcal{H}'$ and $\overline{y} = (y, s) \in \mathcal{H}'$. For $\overline{y} \in \mathcal{H}$, $\dot{G}_{\overline{y}}(\overline{x}) = \dot{G}(\overline{x}, \overline{y})$ is superparabolic on $\mathcal{H}$, parabolic on $\mathcal{H} - \{\overline{y}\}$, and in fact a minimal nonnegative potential supported on $\{\overline{y}\}$ (notice that $\dot{G}_{\overline{y}} \equiv 0$ on $\{\overline{x} = (x, t) | t \leq s\}$). Our first observation is that $\dot{G}$ satisfies a 'proportionality property'.

Lemma 1. *Let X be a complete, noncompact, smooth Riemannian manifold, $\mathcal{H} = X \times (-\infty, 0]$. If u is a potential with support $\{\overline{y}\}$, then $\overline{y} \in \mathcal{H}'$ and u is a constant multiple of $\dot{G}(\overline{x}, \overline{y})$.*

Proof. First observe that, by Lemma 2.1 in [18], which holds for the harmonic space $(\mathcal{H}, P)$, if u is a potential with support $\{\overline{y}\}$, $\overline{y} = (y, s)$, then $u(x, t) = 0$ for $t < s$. Consequently, $u(x, t) \to 0$ as $t \to s$ for any $x \neq y$. If $s = 0$, this implies $u \equiv 0$ on $\mathcal{H}$ by lower semicontinuity, contrary to assumption; hence $\overline{y} \in \mathcal{H}'$.

Let $\overline{y} = (y, s)$. Then $u(x, t) \to 0$ as $t \to s$, $t > s$, for any $x \neq y$. If $U \subset\subset X$ is an open neighborhood of y with smooth boundary and $D = U \times [s, 0]$, define $\phi \in C(\partial_p D)$ by $\phi \equiv 0$ on $U \times \{s\}$, $\phi = u$ on $\partial U \times [s, 0]$. Let

$$v(\overline{x}) = u(\overline{x}) - \omega_D^{\overline{x}}(\phi).$$

Since $u \geq \phi$ on $\partial_p D$ and $u = \phi$ on $\partial_p D - \{\overline{y}\}$, we have $v(\overline{x}) > 0$ and smooth parabolic on $\operatorname{int} D$, and $v(\overline{x}) \to 0$ as $\overline{x} \to \dot{Q} \in \partial_p D - \{\overline{y}\}$. Hence

$$K(x, t) = v(x, t)/v(y, 0)$$

is a parabolic 'kernel function' at (y, s) for the cylinder D, normalized at $(y, 0)$. Since we may assume U is contained in a coordinate neighborhood of X, Theorem 2.7 of [12] (uniqueness of kernel functions for divergence-form time-independent parabolic operators on bounded Lipschitz domains in R^n) implies

$$K(x, t) = \frac{p_U(t-s, x, y)}{p_U(-s, x, y)} \quad (:= K_U(x, t))$$

for $(x, t) \in D$, where p_U denotes the heat kernel of U, vanishing on ∂U.

Taking two potentials u_1 and u_2 with support $\{\overline{y}\}$, this argument shows

$$u_1(\overline{x}) = v_1(y, 0) K_U(t, x, y) + \omega_D^{\overline{x}}(\phi_1),$$

where v_1 and ϕ_1 are obtained from u_1 as above. Thus, normalizing u_2 by

$$\tilde{u}_2 = \frac{v_1(y, 0)}{v_2(y, 0)} u_2,$$

we obtain, for the corresponding $\tilde{\phi}_2$,

$$u_1(\overline{x}) - \tilde{u}_2(\overline{x}) = \omega_D^{\overline{x}}(\phi_1 - \tilde{\phi}_2),$$

which implies $\omega_D^{\overline{x}}(\phi_1 - \tilde{\phi}_2) = 0$, since u_1 and $\tilde{u}_2$ are potentials. This shows $u_1 \equiv \tilde{u}_2$ on D (hence on $\mathcal{H}$), proving that two potentials with support $\{\overline{y}\}$ are proportional. q.e.d.

The construction of the parabolic Martin compactification of $\mathcal{H} = X \times (-\infty, 0]$ follows the lines of the elliptic case (for the parabolic case in $\mathbf{R}^n$, see also [11]), with slight but important changes. We begin by defining the *Martin kernel* with pole $\overline{y} \in \mathcal{H}'$:

$$H_{\overline{y}}(\overline{x}) = H(\overline{x}, \overline{y}) = H(x, t; y, s) = \frac{\dot{G}(\overline{x}, \overline{y})}{\dot{G}(\overline{x}_0, \overline{y})},$$

where $\overline{x}_0 = (x_0, 0)$ and x_0 is a fixed point in X. Let $(\overline{y}_i)_{i \geq 1}$ be a sequence in $\mathcal{H}'$ without accumulation points in $\mathcal{H}'$. Since $H_{\overline{y}_i}(\overline{x}_0) = 1$ for all $i \geq 1$, and $\overline{x}_0$ is 'above' $\mathcal{H}'$, the parabolic Harnack principle implies that some subsequence $(\overline{y}_{i_j})$ converges uniformly on compact sets of $\mathcal{H}'$ to a nonnegative C^2 solution of the heat equation on $\mathcal{H}'$, $H(\overline{x}, \dot{Q}) = H_{\dot{Q}}(\overline{x})$. Define $(\overline{y}_i)_{i \geq 1}$ to be a *fundamental sequence* if either (i) $(\overline{y}_i)$ has no accumulation point in $\mathcal{H}$ and

$$H_{\overline{y}_i}(\overline{x}) \to H_{\dot{Q}}(\overline{x}),$$

a nonnegative C^2 solution of the heat equation, uniformly on compact sets of $\mathcal{H}'$; or (ii) $\overline{y}_i = (y_i, s_i)$, $s_i \to 0$ and (y_i) is bounded in X. In this case we assign to $(\overline{y}_i)$ the trivial Martin kernel $H_{\dot{Q}}(\overline{x}) = \overline{0}$, the identically zero function on $\mathcal{H}$. Notice that if $\overline{y}_i = (y_i, s_i)$ is a sequence on $\mathcal{H}'$ and $s_i \to 0$, $H_{\overline{y}_i}|\mathcal{H}' \to \overline{0}: H_{\overline{y}_i} \equiv 0$ on $X \times (t, 0]$ as soon as $s_i > t$. For all fundamental sequences, $H_{\dot{Q}}$ extends continuously to $\mathcal{H}$, and we have $H_{\dot{Q}} \in P(\mathcal{H})$. Identifying two fundamental sequences if their limiting Martin kernels coincide, we define the *parabolic Martin boundary* of $\mathcal{H}'$ as:

$$\mathcal{P} = \{\text{equivalence classes of fundamental sequences}\},$$

and the parabolic Martin compactification of $\mathcal{H}'$ is

$$\overline{\mathcal{H}} = \mathcal{H} \sqcup \mathcal{P}.$$

As in the harmonic case, we shall frequently identify $\dot{Q} \in \mathcal{P}$ and the corresponding limiting parabolic function $H_{\dot{Q}}(\overline{x})$.

We may endow $\overline{H}$ with a metric in the following way [11]: choose a smooth positive function $0 < \phi < 1$ on $\mathcal{H}$ such that $\int_{\mathcal{H}} \phi(x, t)\, dx\, dt < \infty$, and define

$$d(\overline{y}_1, \overline{y}_2) = \int_{\mathcal{H}} \min\{1, |H_{\overline{y}_1}(\overline{x}) - H_{\overline{y}_2}(\overline{x})|\}\phi(\overline{x})\, dx\, dt$$

for $\overline{y}_1, \overline{y}_2 \in \overline{\mathcal{H}}$. It is easy to check that $(\overline{\mathcal{H}}, d)$ is a compact metric space, and that the topology defined by d coincides with the original topology on $\mathcal{H}'$ (with $\mathcal{H}'$ dense in $\overline{\mathcal{H}}$); $\overline{y}_i \xrightarrow{d} \dot{Q} \in \mathcal{P}$ if and only if $H_{\overline{y}_i} \to H_{\dot{Q}}$ in $\mathcal{H}'$ uniformly on compact sets of $\mathcal{H}'$. Finally, for $\overline{x} \in \mathcal{H}'$, $H_{\overline{x}}(\overline{y}) = H(\overline{x}, \overline{y})$ extends continuously to $\overline{\mathcal{H}} - \{\overline{x}\}$, as a function of $\overline{y}$.

Remark. As just noted, the first major distinction with the harmonic case is that Martin's method allows one to compactify $\mathcal{H}'$, not $\mathcal{H}$. A second important distinction is that, as seen above, the normalization $H(\overline{x}_0, \overline{y}) = 1$ is not always preserved in the limit (it is only lower semi-continuous, as $\overline{y}_i \to \dot{Q} \in \mathcal{P}$).

If X has bounded geometry, one may obtain estimates for the heat kernel of X similar to those in [3]; namely, for each $T > 0$ one has positive constants c, c_1, and $\gamma > \gamma_1$, depending only on T and the bounds on curvature and injectivity radius, such that, for $x, y \in X$ and $t \in (0, T]$:

$$c\frac{e^{-\gamma r^2/t}}{t^{n/2}} \leq p(t, x, y) \leq c_1 \frac{e^{-\gamma_1 r^2/t}}{t^{n/2}},$$

where $r = \operatorname{dist}(x, y)$. The upper estimate is Theorem 4 in [7]. By the main result in [6], the lower bound holds whenever the Ricci curvature of X is bounded below, provided we can verify it for the n-dimensional space form with curvature $-a^2$. For this case, one may use the explicit expressions in [10].

With these estimates for the heat kernel, the argument of Proposition 2.2 in [18] may be used to show that $H_{\overline{y}_i} \to \overline{0}$ in $\mathcal{H}$ if $\overline{y}_i = (y_i, s_i)$, $s_i \to s < 0$ and $y_i \to \infty$ in X.

2. Integral representation of positive parabolic functions

We next obtain a representation formula for positive parabolic functions in $\mathcal{H}$ as an integral of the Martin kernels $H(\overline{x}, \dot{Q})$ with respect to a unique positive Borel measure on the minimal parabolic Martin boundary $\mathcal{P}_1$. This is standard in the case $X = \mathbf{R}^n$ [11, Chapter XIX], and was done in [9, Chapter 11] in the context of abstract potential theory; hence we will just indicate the main steps.

Denote by $\mathcal{S}$ (resp. $\mathcal{L}$, $\mathcal{C}$) the convex cone of nonnegative superparabolic functions on $\mathcal{H}$ (resp. nonnegative potentials, nonnegative parabolic functions). Any $u \in \mathcal{S}$ may be written uniquely as a sum $u = p + h$, where $p \in \mathcal{L}$ and $h \in \mathcal{C}$. $\mathcal{S}$ is a lattice, and may be extended to a vector lattice $[\mathcal{S}]$. As described in [9, 11.2], $[\mathcal{S}]$ may be endowed with a locally convex, separated topology $\mathcal{T}$. $\mathcal{T}$ restricts on $\mathcal{C}$ to the topology defined by uniform convergence on compact subsets of $\mathcal{H}'$. With this topology, $\mathcal{S}$ is a closed convex metrizable cone in $[\mathcal{S}]$ with countable base. Thus, we may appeal to Choquet's theorem to represent points of $\mathcal{S}$ as barycenters of measures.

Due to the normalizations $H_{\dot{Q}}(\overline{x}_0) = 1$ and $H_{\overline{y}}(\overline{x}_0) = 1$, we may only represent 'admissible' superparabolic functions:

$$\hat{\mathcal{S}} = \{u \in \mathcal{S} | u(\overline{x}_0) < \infty\}$$

(with $\hat{\mathcal{L}}$, $\hat{\mathcal{C}} = \mathcal{C}$ defined similarly). Since the evaluation map $\mathcal{S} \times \mathcal{H} \to \mathbf{R}_+ \cup \{\infty\}$ is lower semicontinuous in the $\mathcal{T}$-topology [9, 11.2], $\hat{\mathcal{S}}$ is a closed subcone of $\mathcal{S}$. As observed before, the section $\Sigma = \{u \in \hat{\mathcal{S}} | u(\overline{x}_0) = 1\}$ is not compact; however, the same semicontinuity property implies that

$$\mathcal{K} = \{u \in \hat{\mathcal{S}} | u(\overline{x}_0) \leq 1\}$$

is a *cap* for $\hat{S}$ (i.e., compact, convex, with convex complement); moreover, $\hat{S} = \bigcup_{n>0} nK$. It follows from this that the set of nonzero extremal points of K coincides with the extremal points of Σ $(:= \Sigma_e)$, and Choquet's theorem implies that each point u of $\hat{S}$ is the barycenter of a unique positive Borel measure on Σ_e. Clearly, if u is a potential then the subset of potentials in Σ_e has full measure.

Denote by $\mathcal{L}_e$ $(\mathcal{C}_e, S_e)$ the union of extremal rays in the cone $\hat{\mathcal{L}}$ (resp. $\mathcal{C}$, $\hat{S}$) of admissible potentials (resp. parabolic unctions, admissible superparabolic functions) and by $\mathcal{H}^* = \mathcal{H} \cup \{*\}$ the Alexandroff compactification of $\mathcal{H}$. The support in $\mathcal{H}^*$ of $u \in S_e$ consists of a single point $(= *$ iff $u \in \mathcal{C})$ and the map $\Pi\colon S_e \to \mathcal{H}^*$ so defined is continuous [9, 11.4], and restricts to $\Pi\colon \mathcal{L}_e \to \mathcal{H}'$ (see Lemma 1; in particular, every extremal superparabolic function is 'admissible').

Recall that the *Riesz space* of $\mathcal{H}$ is defined as the quotient space:

$$R_{\mathcal{H}} = \mathcal{L}_e/\text{multiplication by constants.}$$

The continuous map $\Pi\colon \mathcal{L}_e \to \mathcal{H}'$ projects to a map $\Pi'\colon R_{\mathcal{H}} \to \mathcal{H}'$. We will denote by $\langle p \rangle$ the equivalence class in $R_{\mathcal{H}}$ of a potential $p \in \mathcal{L}_e$. The representing measure for a potential on Σ_e projects to a measure on $R_{\mathcal{H}}$, and we obtain: Given an admissible positive potential on $\mathcal{H}$, there exists a unique measure ν on $R_{\mathcal{H}}$ such that

$$u(\overline{x}) = \int_{R_{\mathcal{H}}} H(\overline{x}, \Pi'(\xi))\, d\nu(\xi).$$

To translate this into a representation formula on $\mathcal{H}'$, we need the following lemma.

Lemma 2. *The map $\Pi'\colon R_{\mathcal{H}} \to \mathcal{H}'$ is a homeomorphism, where $R_{\mathcal{H}}$ is given the quotient $\mathcal{T}$-topology.*

Proof. Since Π' is continuous, surjective and injective (by Lemma 1), it is enough to show that it is proper. Take $(y_i, s_i) = \overline{y}_i \to \overline{y} = (y, s) \in \mathcal{H}'$. Then $H_{\overline{y}_i} \in K$ and $\Pi(H_{\overline{y}_i}) = \overline{y}_i$. Assume a subsequence $H_{\overline{y}_{i_j}} \in P(\mathcal{H} - \overline{D})$ for $j \geq j_0(D)$; this implies $H_{\overline{y}_{i_j}} \to p$ uniformly on compact subsets of $\mathcal{H} - \overline{D}$. In particular, support$(p) \subset \{\overline{y}\}$. If support$(p) = \varnothing$ (i.e, $p \in P(\mathcal{H})$), from $p(x, s') = 0$ for any $s' < s$ and all $x \in X$ would follow $p \equiv 0$. This would contradict $H_{\overline{y}_i}(z_i, s_i - s/2) \geq c$ if $d(y_i, z_i) = 1$, where c depends only on $\overline{y}$ (by the lower bound on the heat kernel). Hence support$(p) = \{\overline{y}\}$, and $\langle p \rangle = \langle H_{\overline{y}} \rangle$. This shows that $\langle H_{\overline{y}_{i_j}} \rangle \to \langle p \rangle$, concluding the proof. q.e.d.

From this lemma and the representation formula preceding it, we obtain the Riesz representation theorem for parabolic potentials: for any admissible potential $p(\overline{x})$ in $\mathcal{H}$, there exists a unique finite Borel measure μ on $\mathcal{H}'$ such that

$$p(\overline{x}) = \int_{\mathcal{H}'} H(\overline{x}, \overline{y})\, d\mu(\overline{y}), \qquad \mu(\mathcal{H}') = p(\overline{x}_0).$$

The existence part of the Martin representation theorem follows from this proposition essentially by the same method as in the elliptic case. Define the *reduction* of a nonnegative superparabolic function u over a set $D \subset \mathcal{H}$ by:

$$R^u_D(x) = \inf\{v \in S \mid v \geq u \text{ on } D\},$$

and the *balayage* $\hat{R}^u_D(x)$ as the lower semicontinuous regularization of R^u_D. Then $\hat{R}^u_D$ is superparabolic, $\hat{R}^u_D = u$ on D if D is open, and on $\mathcal{H} - \overline{D}$, $\hat{R}^u_D$ equals R^u_D and is parabolic. The main property of $\hat{R}^u_D$ used in the proof of the Martin representation theorem is that if D is compact in $\mathcal{H}$, $\hat{R}^u_D$ is a potential. This is easy to see if u is parabolic: since u is bounded on D, $u(\overline{x}) \leq M\dot{G}(\overline{x}, \overline{y})$, $\overline{x} \in D$, for any $\overline{y}$ 'below' D, and so $\hat{R}^u_D \leq M\dot{G}_{\overline{y}}$ on $\mathcal{H}$, hence is a potential. Thus, if u is a nonnegative parabolic function on $\mathcal{H}$, taking an exhaustion of $\mathcal{H}$ by cylinders D_i containing $\overline{x}_0$, whose closure in $\mathcal{H}$ is compact, and applying the representation formula for potentials, we get:

$$\hat{R}^u_{D_i}(\overline{x}) = \int_{\mathcal{H}'} H(\overline{x}, \overline{y})\, d\mu_i(\overline{y}), \qquad \mu_i(\mathcal{H}') = u(\overline{x}_0).$$

In fact, since $\hat{R}^u_{D_i}$ is parabolic on D_i and on $\mathcal{H} - \overline{D}_i$, it follows that support$(\mu_i) \subset \partial_p D_i$. Since

$$\hat{R}_{D_i}(\overline{x}) \uparrow \hat{R}^u_{\mathcal{H}}(\overline{x}) = u,$$

passing to a subsequence and taking a weak limit of the measures μ_i we obtain the existence part of the Martin representation theorem: there exists a finite Borel measure μ on $\mathcal{P}$ such that

$$u(\overline{x}) = \int_{\mathcal{P}} H(\overline{x}, \dot{Q})\, d\mu(\dot{Q}), \qquad \mu(\mathcal{P}) = u(\overline{x}_0).$$

Now let $u(\overline{x})$ be a *minimal* nonnegative parabolic function on $\mathcal{H}$. Then for some $\dot{Q} \in \mathcal{P}$, $H_{\dot{Q}} \neq \overline{0}$, we have

$$u(\overline{x}) = (\text{const}) H_{\dot{Q}}(\overline{x}).$$

This follows immediately from the preceding representation formula and the fact that, by minimality, any measure representing u on $\mathcal{P}$ must consist of a single atom $\{\dot{Q}\}$.

As in the elliptic case, one always has a unique representing measure on the *minimal parabolic Martin boundary*:

$$\mathcal{P}_1 = \{0 \le u \in P(\mathcal{H}) | u(\overline{x}_0) = 1 \text{ and } 0 \le v \le u,\ v \in P(\mathcal{H}) \Rightarrow v = (\text{const})u\}.$$

As seen above, $\mathcal{P}_1$ may be identified with a subset of $\mathcal{P}$.

To obtain the unique representation theorem on $\mathcal{P}_1$, one may either proceed directly (as in [11]), or appeal to Choquet theory, as explained above in the case of potentials; Theorem 11.3.1 in [9] implies that any $u \in \mathcal{C}$ is the barycenter of a unique measure μ_u on the section $\mathcal{P}_1$ of the extremal rays of $\mathcal{C}$:

Parabolic Martin representation theorem. *Let $u(x, t) \ge 0$, $u \in P(\mathcal{H})$. There exists a unique finite Borel measure μ_u on $\mathcal{P}_1$ such that*

$$u(x, t) = \int_{\mathcal{P}_1} H(x, t; \dot{Q})\, d\mu_u(\dot{Q})$$

for $(x, t) \in \mathcal{H}$, and

$$\mu_u(\mathcal{P}_1) = u(x_0, 0).$$

Under a mild geometric assumption, a theorem of A. Korányi and J. C. Taylor [15] gives a complete characterization of the minimal positive parabolic functions. As before, let X be complete noncompact with bounded geometry, $\mathcal{H} = X \times (-\infty, 0]$.

Proposition 1. *Assume X has Ricci curvature bounded below. Then*

$$\mathcal{P}_1(\mathcal{H}) = \{\dot{Q} \in \mathcal{P} | H_{\dot{Q}}(x, t) = e^{\lambda t} K^{\lambda}(x, Q),\ \lambda \ge \lambda_0(X),\ Q \in \mathcal{M}_1^{\lambda}(X)\}.$$

Proof. (i) The fact that any minimal positive parabolic function $u(x, t)$ normalized at $(x_0, 0)$ has the above form was proved in [15]. For completeness, we sketch the idea of a slightly different proof. For $s > 0$, let $u_s(x, t) = u(x, t - s)$. Then $u_s \in P(\mathcal{H})$, and by the parabolic Harnack inequality of [15],

$$u_s(x, t) \le c_s u(x, t) \quad \text{for } (x, t) \in \mathcal{H},$$

where c_s depends only on s and the lower curvature bound of X. Since u is minimal, this implies

$$u_s(x, t) = u(x_0, -s) u(x, t).$$

Letting $0 \le \phi(t) = u(x_0, -t) \in C([0, \infty))$, it follows that $\phi(t + s) = \phi(t)\phi(s)$ for all $s, t \ge 0$, and $\phi(0) = 1$, so $\phi(t) = e^{\lambda t}$. This implies $u(x, t - s) = e^{-\lambda s} u(x, t)$, $(x, t) \in \mathcal{H}$, $s \ge 0$, so

$$u(x, t) = e^{-\lambda t} u(x, 0).$$

Letting $u_0(x) = u(x, 0)$, one has $\Delta u_0 = \lambda u_0$, so $\lambda \geq \lambda_0$. The minimality of $u(x, t)$ clearly implies that of u_0, so $u_0 \in \mathcal{M}_1^\lambda(X)$.

Remark. Part (i) of the proof allows us to define the continuous map:

$$p: \mathcal{P}_1 \to [\lambda_0(X), \infty), \dot{Q} \mapsto \lambda \text{ such that}$$
$$H(x, t; \dot{Q}) = e^{\lambda t}\phi(x), \quad \text{where } \phi \in \mathcal{M}_1^\lambda(X).$$

In fact p is the restriction of the continuous map $p: \mathcal{P} \to [\lambda_0(X), +\infty)$ defined by $p(u) = -\log(u(x_0, -1)/u(x_0, 0))$.

(ii) One may also show that, conversely, each positive parabolic function having the form

$$u(x, t) = e^{\lambda t}\phi_\lambda(x),$$

where $\phi_\lambda \in \mathcal{M}_1^\lambda$, is minimal parabolic.

In the following proof, we denote by $\{p(\dot{Q}) > \lambda\}$ the subset $p^{-1}((\lambda, \infty))$ of $\mathcal{P}_1$. Given the equality

$$\int_{\mathcal{P}_1} H(x, t, \dot{Q})\, d\mu(\dot{Q}) = e^{\lambda t}\phi_\lambda(x)$$

for a probability μ on $\mathcal{P}_1$, we must show that μ consists of a single atom $\dot{Q}$, such that $H_{\dot{Q}} = e^{\lambda t}\phi_\lambda(x)$. First notice that $\mu(\{p(\dot{Q}) < \lambda_1\}) = 0$ for any $\lambda_1 < \lambda$, since evaluating the above equality at (x_0, t) gives:

$$\mu(\{p(\dot{Q}) < \lambda_1\})e^{\lambda_1 t} \leq \int_{\{p(\dot{Q})<\lambda_1\}} e^{p(\dot{Q})t}\, d\mu(\dot{Q}) \leq e^{\lambda t} < e^{\lambda_1 t}$$

for arbitrary $t < 0$. Thus,

$$1 = \phi_\lambda(x_0) = \int_{\{p(\dot{Q})\geq\lambda\}} e^{-\lambda t}e^{p(\dot{Q})t}\, d\mu(\dot{Q}),$$

or

$$1 - \int_{\{p(\dot{Q})=\lambda\}} e^{-(\lambda-p(\dot{Q}))t}\, d\mu(\dot{Q}) = \int_{\{p(\dot{Q})>\lambda\}} e^{-(\lambda-p(\dot{Q}))t}\, d\mu(\dot{Q}).$$

Since the right-hand side is decreasing as $t \to -\infty$ and the left-hand side is constant in t, both must vanish. This implies $\mu(\{p(\dot{Q}) = \lambda\}) = 1$. Denoting by ν the probability $\nu = (b_\lambda)_*(\mu_{|\{p(\dot{Q})=\lambda\}})$ on $\mathcal{M}_1^\lambda$, we have

$$e^{\lambda t}\phi_\lambda(x) = \int_{\mathcal{M}_1^\lambda} e^{\lambda t}K^\lambda(x, Q)\, d\nu(Q)$$

for $t \leq 0$. Setting $t = 0$, the minimality of ϕ_λ shows that ν must consist of a single atom $\{Q\}$, such that $\phi_\lambda = K_Q^\lambda$. This concludes the proof.

3. A representation formula for harmonic functions

The proof of the main result starts by considering the Martin representation of a positive parabolic function f on X (complete, with Ric bounded below) as a positive harmonic function on $\mathcal{H}$. Assuming $f(x_0) = 1$, we have

$$f(x) = \int_{\mathcal{P}_1} H(x, t, \dot{Q})\, dm(\dot{Q}) \tag{1}$$

for a unique Borel probability m on $\mathcal{P}_1$. It is natural to expect that m gives full measure to the set of $\dot{Q}$ whose kernel is independent of t. This may be proved by an argument similar to that in [20]. Denote

$$\mathcal{P}_1^0 = \{\dot{Q} \in \mathcal{P}_1 | p(\dot{Q}) = 0\}$$

(the map $p: \mathcal{P}_1 \to [\lambda_0(X), +\infty)$ was defined in §2).

Proposition 2. *If $f > 0$ is harmonic on X, the probability m in* (1) *satisfies $m(\mathcal{P}_1 - \mathcal{P}_1^0) = 0$.*

Proof. Denote

$$\mathcal{P}^+ = p^{-1}((0, \infty)) \subset \mathcal{P}_1,$$
$$\mathcal{P}_\lambda^- = p^{-1}((\lambda_0(X), \lambda)) \subset \mathcal{P}_1 \quad \text{for } \lambda < 0.$$

Then

$$\int_{\mathcal{P}_\lambda^-} H(x, t, \dot{Q})\, dm(\dot{Q}) \le f(x).$$

By Proposition 1, this inequality evaluated at (x_0, t), $t \le 0$, gives

$$\int_{\mathcal{P}_\lambda^-} e^{p(\dot{Q})}\, dm(\dot{Q}) \le f(x_0).$$

If $t \le 0$, the left-hand side is bounded below by $m(\mathcal{P}_\lambda^-)e^{\lambda t}$. Letting $t \to -\infty$ shows that $m(\mathcal{P}_\lambda^-) = 0$ for any $\lambda < 0$. Thus,

$$f(x) - \int_{\mathcal{P}_1^0} H(x, t, \dot{Q})\, dm(\dot{Q}) = \int_{\mathcal{P}^+} H(x, t, \dot{Q})\, dm(\dot{Q}).$$

As $t \uparrow 0$, $t < 0$, evaluating the left-hand side at x_0 gives a constant, the right-hand side an increasing function of t. This shows that $m(\mathcal{P}^+) = 0$, concluding the proof.

Proof of the main theorem, part (i). In the following, for each factor X_a, $a = 1, 2$, we adopt the notation $\mathcal{H}_a = X_a \times (-\infty, 0]$, with coordinates (x^a, t), $\mathcal{P}^a$ is the parabolic Martin boundary of $\mathcal{H}_a$, where we normalize the Martin kernels $H_a(x^a, t; y^a, s)$ and $H_a(x^a, t; \dot{Q}_a)$, $\dot{Q}_a \in \mathcal{P}^a$, at $(x_0^a, 0)$, and $\mathcal{P}_1^a$ is the minimal parabolic Martin boundary of $\mathcal{H}_a$.

Let f be a positive harmonic function on X, $f(x_0) = 1$. By Proposition 2, f is represented by a unique probability on $\mathcal{P}_1^0$:

$$f(x) = \int_{\mathcal{P}_1^0} H(x, t, \dot{Q})\, dm(\dot{Q}), \tag{2}$$

where the Martin kernels in $\mathcal{P}_1^0$ are independent of t.

Claim. If $\dot{Q} \in \mathcal{P}_1^0$, then

$$H(x, t, \dot{Q}) = K^{\lambda_1}(x_1, Q_1) K^{\lambda_2}(x_2, Q_2),$$

with $\lambda_a \geq \lambda_0(X_a)$, $\lambda_1 + \lambda_2 = 0$, and $Q_a \in M_1^{\lambda_a}(X_a)$.

The main observation is that the heat kernel of X is given by

$$p(t, x, y) = p_1(t, x^1, y^1) p_2(t, x^2, y^2),$$

where $p_a(t, x^a, y^a)$ is the heat kernel of X_a. This implies a similar splitting for the parabolic Martin kernels with pole at $(y, s) \in \mathcal{H}'$:

$$H(x, t; y, s) = H_1(x^1, t; y^1, s) H_2(x^2, t; y^2, s). \tag{3}$$

Now take a fundamental sequence $(y_i, s_i) \to \dot{Q} \in \mathcal{P}$. By taking subsequences, we may assume $(y_i^1, s_i) \to \dot{Q}_1 \in \mathcal{P}^1$ and $(y_i^2, s_i) \to \dot{Q}_2 \in \mathcal{P}^2$. Then (3) implies

$$H(x, t, \dot{Q}) = H_1(x^1, t, \dot{Q}_1) H_2(x^2, t, \dot{Q}_2). \tag{4}$$

If $\dot{Q} \in \mathcal{P}_1$, it follows easily that any $\dot{Q}_1 \in \mathcal{P}^1$, $\dot{Q}_2 \in \mathcal{P}^2$ satisfying (4) are also minimal; thus, by Proposition 1

$$H(x, t, \dot{Q}) = e^{\lambda_1 t} K^{\lambda_1}(x^1, Q_1) e^{\lambda_2 t} K^{\lambda_2}(x^2, Q_2),$$

with $\lambda_a \geq \lambda_0(X_a)$ and $Q_a \in \mathcal{M}_1^{\lambda_a}(X_a)$ (since this expression implies easily that the K^{λ_a} are minimal in $\mathcal{M}^{\lambda_a}(X_a)$ if $H_{\dot{Q}} \in \mathcal{P}_1$). In particular, $p(\dot{Q}) = 0$ implies $\lambda_1 + \lambda_2 = 0$, and we have

$$H(x, t, \dot{Q}) = K^{\lambda_1}(x^1, Q_1) K^{\lambda_2}(x^2, Q_2)$$

for any $\dot{Q} \in \mathcal{P}_1^0$, proving the claim.

If f is *minimal* positive harmonic, the support of the measure m in (2) must consist of a unique $\dot{Q} \in \mathcal{P}_1^0$, so that

$$f(x) = H_{\dot{Q}}(x, t) = K^{\lambda_1}(x^1, Q_1) K^{\lambda_2}(x^2, Q_2).$$

4. Minimal positive harmonic functions

Before proving part (ii) of the main result, we recall some results in potential theory. We preserve the notation of the preceding sections. For any $\lambda > \lambda_0(X)$ and any relatively compact open set $U \subset X$ with smooth boundary, one may solve the Dirichlet problem for $(\Delta - \lambda I)$ and continuous data on ∂U. For the (Brelot) harmonic space defined by $(\Delta - \lambda I)$, a continuous function f on X is λ-subharmonic if $f(x) \leq \omega_\lambda^x(f_{|\partial U})$ for any λ-regular open set $U \subset\subset X$, where ω_λ^x denotes λ-harmonic measure on ∂U. Clearly, if a sequence $(f_k)_{k\geq 1}$ of continuous λ-subharmonic functions on X converges uniformly on compact sets of a continuous function f, f is also λ-subharmonic. It is well known that a smooth function f on X is λ-subharmonic iff $\Delta f \geq \lambda f$.

Let $X = X_1 \times X_2$ be a Riemannian product, where X_1 and X_2 are complete noncompact Riemannian manifolds whose Ricci curvature is bounded below. For $\lambda > \lambda_0(X_1)$, define

$$E_\lambda^1 = \{v \in P(\mathcal{H}) | v \geq 0 \text{ and } v(-1, x_1, x_2^0) \text{ is } \lambda\text{-subharmonic on } X_1\},$$
$$F_\lambda^1 = \{v \in P(\mathcal{H}) | v \geq 0 \text{ and } \Delta_1 v(-1, x_1, x_2^0) = \lambda v(-1, x_1, x_2^0)\}.$$

Then E_λ^1 and F_λ^1 are convex subcones of $\mathcal{C} = \{v \in P(\mathcal{H}) | v \geq 0\}$, closed under uniform convergence on compact subsets of $\mathcal{H}'$. Clearly $E_\lambda^1 \subset E_\mu^1$ if $\lambda > \mu$. E_λ^2 and F_λ^2 are similarly defined for the factor X_2 (and $\lambda > \lambda_0(X_2)$); denote $F_{\lambda_1, \lambda_2} = F'_{\lambda_1} \cap F_{\lambda_2}^2$. If $\lambda_1 + \lambda_2 = 0$, Proposition 1 implies

$$\mathcal{P}_1^0 = \bigcup_{\substack{\lambda_n \to \lambda_1 \\ \lambda_n > \lambda_1}} (E_{\lambda_n}^1 \cap \mathcal{P}_1^0) \amalg (F_{\lambda_1, \lambda_2} \cap \mathcal{P}_1^0) \amalg \bigcup_{\substack{\mu_n \to \lambda_2 \\ \mu_n > \lambda_2}} (E_{\mu_n}^2 \cap \mathcal{P}_1^0).$$

The proof of part (ii) is based on the following lemma, due to Cheng and Yau.

Lemma 3. *Let X be a complete Riemannian manifold with* $\mathrm{Ric}(X) \geq K$

(a) [8, *Theorem* 6] *If $f \in C^2(X)$, $f > 0$, and $\Delta f = \lambda f$ on X, then* $|\Delta \log f| \leq c_n \max\{|\lambda|, |K|\}$.

(b) [8, *Theorem* 3] *Let $f \in C^2(X)$; suppose f is bounded above and does not attain its supremum on X. Then $\exists x_k \to \infty$ on X such that* $f(x_k) \to \sup f$ *and*

(c) $|\nabla f|(x_k) \leq c_1/r_k$,

(d) $\Delta f(x_k) \leq c_2/r_k$,

where $r_k = d(x_k, x_0)$, c_1 *depends only on* $\dim X$ *and* $\sup f$, *and* c_2 *depends additionally on* K.

Proof of the main theorem, part (ii). Let $v > 0$ be harmonic on X, $v(x_0) = 1$. Assume $v \leq K^{\lambda_1}(x_1)K^{\lambda_2}(x_2)$ (where $K^{\lambda_i} \in \mathcal{M}_1^{\lambda_i}(X_i)$ and $\lambda_1 + \lambda_2 = 0$) and denote by m_v the representing probability on $\mathcal{P}_1^0$:

$$v = \int_{\mathcal{P}_1^0} H_{\dot{Q}}\, dm_v(\dot{Q}).$$

Claim. For any $\overline{\lambda}_1 > \lambda_1$, $m_v(E^1_{\overline{\lambda}_1} \cap \mathcal{P}_1^0) = 0$.

Proof. If not, let

$$w = \int_{E^1_{\overline{\lambda}_1} \cap \mathcal{P}_1^0} H_{\dot{Q}}\, dm_v(\dot{Q}).$$

Then $0 < w \leq K^{\lambda_1}K^{\lambda_2}$ and $w \in E^1_{\overline{\lambda}_1}$, so defining $f(x_1) = w(-1, x_1, x_2^0)$, we have $f \in C^\infty(X_1)$, $0 < f \leq K^{\lambda_1}$ and $\Delta_1 f \geq \overline{\lambda}_1 f$. Now let $h(x_1) = f/K^{\lambda_1}$. Then h satisfies $0 < h \leq 1$ and

$$\begin{aligned}\Delta_1 h &= \frac{\Delta_1 f}{K^{\lambda_1}} + 2\nabla_1 h \cdot \nabla_1 \log K^{\lambda_1} - h\frac{\Delta_1 K^{\lambda_1}}{K^{\lambda_1}} \\ &\geq (\overline{\lambda}_1 - \lambda_1)h - 2|\nabla_1 h|\,|\nabla_1 \log K^{\lambda_1}| \\ &\geq (\overline{\lambda}_1 - \lambda_1)h - c|\nabla_1 h|,\end{aligned}$$

by part (a) of Lemma 3.

By part (b) of Lemma 3, $\exists x_k \to \infty$ on X such that $h(x_k) \to \sup h$ (the preceding inequality implies h cannot have a positive interior maximum). $|\nabla_1 h|(x_k) \leq c_1/r_k$ and $\Delta_1 h(x_k) \leq c_2/r_k$, so (by the inequality above) $h(x_k) \to 0$. This shows $h \equiv 0$, a contradiction. Analogously, $m_v(E^2_{\overline{\lambda}_2} \cap \mathcal{P}_1^0) = 0$ for any $\overline{\lambda}_2 > \lambda_2$. Thus,

$$v = \int_{F_{\lambda_1, \lambda_2} \cap \mathcal{P}_1^0} H_{\dot{Q}}\, dm_v(\dot{Q}),$$

which implies $\Delta_2 v = \lambda_2 v$. Since $v(x_1^0, x_2) \leq K^{\lambda_2}(x_2)$, the minimality of K^{λ_2} gives $v(x_1^0, x_2) = K^{\lambda_2}(x_2)$. On the other hand, since $\Delta_1 v = \lambda_1 v$, $v \leq K^{\lambda_1}K^{\lambda_2}$, and K^{λ_1} is minimal,

$$\frac{v(x_1, x_2)}{K^{\lambda_2}(x_2)} = \frac{v(x_1^0, x_2)}{K^{\lambda_2}(x_2)} K^{\lambda_1}(x_1),$$

so $v(x_1, x_2) = K^{\lambda_1}(x_1)K^{\lambda_2}(x_2)$, concluding the proof.

5. Bounded harmonic functions and Poisson boundary

In this section we indicate how the minimality of product kernels implies a strong harmonicity theorem for bounded harmonic functions.

Let X be a complete Riemannian manifold and f a positive harmonic function on X. The unique measure μ_f representing f on the minimal Martin boundary

$$f(x) = \int_{\mathcal{M}_1} K(x, Q)\, d\mu_f(Q)$$

satisfies the following monotonicity property:

$$\Delta f = \Delta g = 0, \quad 0 < f \leq g \Rightarrow \mu_f \prec \mu_g.$$

This follows immediately from the fact that, for any Borel set $A \subset \mathcal{M}$,

$$\mu_f(A) = R_A^f(x_0),$$

where the reduction R_A^f of f over A is defined as the infimum of $\hat{R}_U^f$ over all traces U in X of open neighborhoods of A in $\overline{X}$, with $\hat{R}_U^f$ defined as in §2 (see [19] or [11, Chapter XII]).

This property allows one to obtain a representation formula for bounded harmonic functions. Denote by ω the unique probability on $\mathcal{M}_1$ representing the constant function 1 on X ('harmonic measure' on $\mathcal{M}_1$), and by $\Pi(X) \subset \mathcal{M}$ the closed support of ω.

Definition. The *Poisson boundary* of X is the pair $(\Pi(X), \omega)$.

Let f be a bounded harmonic function on X (which we may assume positive by adding a constant). Since $\mu_f \prec \omega$, we have $\phi \in L^1(\Pi(X), \omega)$, ω-a.e. uniquely defined by f, such that

$$f(x) = \int_{\Pi(X)} K(x, Q)\phi(Q)\, d\omega(Q).$$

Now consider a Riemannian product $X = X_1 \times X_2$, as in the main theorem. By part (ii), if $K_a(x^a, Q_a) \in \mathcal{M}_1(X_a)$, the product

$$f(x) = K_1(x^1, Q_1)K_2(x^2, Q_2)$$

is a minimal positive harmonic function on X. This gives a continuous injective map

$$I: \mathcal{M}_1(X_1) \times \mathcal{M}_1(X_2) \to \mathcal{M}_1(X).$$

Denote by ω_a the probability on $\mathcal{M}_1(X_a)$ representing the constant function 1 on X_a:

$$1 = \int_{\mathcal{M}_1(X_a)} K(x^a, Q_a)\, d\omega_a(Q_a).$$

Clearly, for the constant function 1 on X we have the representation formula

$$1 = \int_{\mathcal{M}_1(X_1)\times\mathcal{M}_1(X_2)} K(x^1, Q_1)K(x^2, Q_2)\, d\omega_1(Q_1)\times d\omega_2(Q_2),$$

so that the (unique) measure ω representing 1 on $\mathcal{M}_1(X)$ is

$$\omega = I_*(\omega_1 \times \omega_2).$$

Thus, given a bounded harmonic function f on X, we have $\phi(Q) \in L^1(\mathcal{M}_1, \omega)$ (ω-a.e. uniquely defined by f) such that

$$\begin{aligned} f(x) &= \int_{\mathcal{M}_1} K(x, Q)\phi(Q)\, dI_*(\omega_1\times\omega_2)(Q) \\ &= \int_{\mathcal{M}_1^1\times\mathcal{M}_1^2} K(x^1, Q_1)K(x^2, Q_2)\psi(Q_1, Q_2)\, d\omega_1\times d\omega_2(Q_1, Q_2), \end{aligned}$$

where $\psi = \phi\circ I$. It follows immediately from this formula that $\Delta_a f = 0$, $a = 1, 2$.

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